Appendix D

Analysis of CD Learning: Auxiliary Derivations

The appendix offers some of the auxiliary derivations supporting the result (1.33) for the second and fourth moments $\langle x_i^2 \rangle$ and $\langle x_i^4 \rangle$ of the i^{th} reconstruction of the visible variable x_i for the *fixed* initial point x_0 [see section 1.4.3].

☐ To express $\langle x_i^2 \rangle$ through the initial data point x_0 we make use of the recurrent relation (1.30):

$$\begin{cases} z_i = \sigma x_i + \tau u_i, \ i \geq 0 \\ x_j = \sigma z_{j-1} + \tau v_j, \ j \geq 1. \end{cases} \tag{D.1}$$

From (D.1) we can easily see that

$$z_0 = x_0\sigma + \tau u_0 \tag{D.2}$$

$$x_1 = x_0\sigma^2 + \tau u_0\sigma + \tau v_1 \tag{D.3}$$

$$z_1 = x_0\sigma^3 + \tau u_0\sigma^2 + \tau v_1\sigma + \tau u_1 \tag{D.4}$$

$$x_2 = x_0\sigma^4 + \tau u_0\sigma^3 + \tau v_1\sigma^2 + \tau u_1\sigma + \tau v_2 \tag{D.5}$$

$$\ldots$$

$$z_2 = x_0\sigma^5 + \tau u_0\sigma^4 + \tau v_1\sigma^3 + \tau u_1\sigma^2 + \tau v_2\sigma + \tau u_2 \quad (D.6)$$

$$x_3 = x_0\sigma^6 + \tau u_0\sigma^5 + \tau v_1\sigma^4 + \tau u_1\sigma^3 + \tau v_2\sigma^2 + \tau u_2\sigma + \tau v_3 \quad (D.7)$$

Continuing in the similar manner we can notice that

$$\begin{aligned} x_i &= x_0\sigma^{2i} + (\tau u_0)\sigma^{2i-1} + (\tau v_1)\sigma^{2i-2} + \ldots + (\tau u_{i-1})\sigma + \tau v_i\sigma^0 \\ &= x_0\sigma^{2i} + \tau\sum_{k=0}^{i-1} u_k\sigma^{2i-1-2k} + \tau\sum_{k=1}^{i} v_k\sigma^{2i-2k} \\ &= x_0\sigma^{2i} + \tau\sum_{k=1}^{i}\sigma^{2i-2k}(u_{k-1}\sigma + v_k). \end{aligned} \quad (D.8)$$

Then by taking a square of (D.8) we see that

$$\begin{aligned} x_i^2 &= [x_0\sigma^{2i} + \tau\sum_{k=1}^{i}\sigma^{2i-2k}(\sigma u_{k-1} + v_k)]^2 \\ &= \sigma^{4i}[x_0^2 + 2x_0\tau\sum_{k=1}^{i}C_k\sigma^{-2k} + \tau^2(\sum_{k=1}^{i}\sigma^{-2k}C_k)^2], \end{aligned} \quad (D.9)$$

where

$$C_k = \sigma u_{k-1} + v_k. \quad (D.10)$$

Since $u_k,\ v_k \sim N(0,1)$ we obtain $\langle C_k\rangle = 0$ and

$$\langle x_i^2\rangle = \sigma^{4i}\left[\langle x_0^2\rangle + \tau^2\langle(\sum_{k=1}^{i}\sigma^{-2k}C_k)^2\rangle\right]. \quad (D.11)$$

Considering that $\langle u_i v_k\rangle = 0,\ \langle u_i u_k\rangle = \langle v_i v_k\rangle = \delta_{ik}$, where δ_{ik} is the Kronecker delta, and performing some algebraic manipulations with (D.11), we can derive that

$$\langle x_i^2\rangle = \sigma^{4i}\left[\langle x_0^2\rangle + (\sigma^2+1)\tau^2\sum_{k=1}^{i}\sigma^{-4k}\right]. \quad (D.12)$$

Notice that the sum of the geometric series in the term in brackets of the r.h.s. of (D.12) can be written as

$$\sum_{k=1}^{i}\sigma^{-4k} = \frac{\sigma^{-4}(1-\sigma^{-4i})}{1-\sigma^{-4}} = \frac{\sigma^{4i}-1}{\sigma^{4i}(\sigma^4-1)}. \quad (D.13)$$

Substituting this expression into (D.12) we obtain

$$\begin{aligned}\langle x_i^2\rangle &= \sigma^{4i}x_0^2+\frac{\sigma^{4i}(\sigma^2+1)\tau^2(\sigma^{4i}-1)}{\sigma^{4i}(\sigma^4-1)}\\ &= \sigma^{4i}x_0^2+\frac{\tau^2(\sigma^{4i}-1)}{\sigma^2-1}\\ &= \sigma^{4i}(x_0^2-a)+a, \end{aligned}\tag{D.14}$$

where $\tau^2 \stackrel{\text{def}}{=} a(1-\sigma^2)$.

In order to estimate $\langle x_i^4\rangle$ we will first notice that if u_k ∼N(0,1) then $\langle u_k\rangle = 0$, $\langle u_k^2\rangle = 1$, $\langle u_k^3\rangle = 0$, and $\langle u_k^4\rangle = 3$[1]. Then (D.10) $\Rightarrow$ $\langle C_k\rangle = 0, \langle C_k^2\rangle = \sigma^2+1, \langle C_k^3\rangle = 0, \langle C_k^4\rangle = 3(\sigma^2+1)$.

From (D.9) we find that

$$x_i^4 = \sigma^{8i}(x_0^4+6x_0^2\tau^2A^2+\tau^4A^4+4x_0^3\tau A+4x_0\tau^3A^3), \tag{D.16}$$

where

$$A=\sum_{k=1}^{i}\sigma^{-2k}C_k. \tag{D.17}$$

Due to the condition of independence $I(C_k, C_i)$ for $k \neq i$ we easily derive

$$\langle A\rangle = \langle A^3\rangle = 0 \tag{D.18}$$

$$\langle A^2\rangle = \sum_{k=1}^{i}\sigma^{-4k}(\sigma^2+1), \tag{D.19}$$

what leads to

$$\langle x_i^4\rangle = \sigma^{8i}(x_0^4-6x_0^2a\frac{\sigma^{4i}-1}{\sigma^{4i}}+\tau^4\langle A^4\rangle). \tag{D.20}$$

[1] We can easily show this either by considering the relation for higher-order moments of Gaussians given by (Chacon and Iranpour, 1988)

$$\begin{aligned}\langle x_1x_2x_3x_4\rangle &= \langle x_1x_2\rangle\langle x_3x_4\rangle\\ &+ \langle x_1x_3\rangle\langle x_2x_4\rangle+\langle x_1x_4\rangle\langle x_2x_3\rangle, \end{aligned}\tag{D.15}$$

or by computing derivatives of the moment-generating function of a normalized Gaussian $\partial^n e^{\frac{t^2}{2}}/\partial t^n|_{t=0}$.

In order to estimate $\langle A^4 \rangle$ we notice that

$$A^4 = \left(\sum_{k=1}^{i} \sigma^{-4k} C_k^2\right)^2 + 4\sum_{k=1}^{i} \sigma^{-4k} C_k^2 \sum_{f=1}^{i-1} \sum_{l=f+1}^{i} C_f C_l \sigma^{-2f} \sigma^{-2l} \tag{D.21}$$
$$+ \; 4\left(\sum_{f=1}^{i-1} \sum_{l=f+1}^{i} C_f C_l \sigma^{-2f} \sigma^{-2l}\right)^2,$$

and the expectation of the second term in the r.h.s. of the equality is zero. Expanding (D.21) it can then be shown that

$$\begin{aligned} \langle A^4 \rangle &= \left\langle \sum_{k=1}^{i} \sigma^{-8k} C_k^4 \right\rangle + 6 \left\langle \left(\sum_{f=1}^{i-1} \sum_{l=f+1}^{i} C_f C_l \sigma^{-2f} \sigma^{-2l}\right)^2 \right\rangle \\ &= \sum_{k=1}^{i} \sigma^{-8k} \cdot 3(1+\sigma^2)^2 + 6(1+\sigma^2)^2 \sum_{f=1}^{i-1} \sigma^{-4f} \sum_{l=f+1}^{i} \sigma^{-4l} \\ &= 3(1+\sigma^2)^2 \frac{(\sigma^{8i}-1)}{\sigma^{8i}} \cdot \frac{1}{\sigma^8 - 1} + 6(1+\sigma^2)^2 \frac{1}{\sigma^{8i}(\sigma^4-1)} \left[\frac{\sigma^{8i}-1}{\sigma^8-1} - \frac{\sigma^{4i}-1}{\sigma^4-1}\right]. \end{aligned} \tag{D.22}$$

Substituting this expression into (D.20), we obtain the expression for $\langle x_i^4 \rangle$:

$$\begin{aligned} \langle x_i^4 \rangle &= x_0^4 \sigma^{8i} + 3a^2 \frac{(\sigma^4-1)(\sigma^{8i}-1)}{\sigma^4+1} \\ &+ \; 6a^2(\sigma^4-1)^2 \left[\frac{\sigma^{8i}-1}{\sigma^8-1} - \frac{\sigma^{4i}-1}{\sigma^4-1}\right] - 6ax_0^2\sigma^{4i}(\sigma^{4i}-1). \end{aligned} \tag{D.23}$$

Notice that if we estimated $\langle x_i^4 \rangle$ according to relation (D.15) for higher-order Gaussian moments, we would get $\langle x_i^4 \rangle^\star = 3(\langle x_i^2 \rangle)^2$, what would result in $\langle x_0^4 \rangle = 3(\langle x_0^2 \rangle)^2$, yielding $\langle x_0^4 \rangle = 3x_0^4$ for a fixed initial point x_0. Clearly, this overestimates expectation of the fixed value $\langle x_0^4 \rangle = x_0^4$, and in order to obtain a correct estimate of $\langle x_i^4 \rangle$ from $\langle x_i^4 \rangle^\star$ we would need to reduce $\langle x_i^4 \rangle^\star$ by a factor of $2x_0^4$. In our case this results in

$$\begin{aligned} \langle x_i^4 \rangle &= \langle x_i^4 \rangle^\star - 2x_0^4\sigma^{8i} \\ &= 3(\sigma^{4i}(x_0^2 - a) + a)^2 - 2x_0^4\sigma^{8i}. \end{aligned} \tag{D.24}$$

This is identical to the previously obtained relation (D.23).

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